Exactly soluble isotropic spin-$\frac{1}{2}$ ladder models

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Abstract
The undeformed limit of the dilute two-colour braid–monoid algebra gives a natural basis for the description of spin-$\frac{1}{2}$ ladder models, and allows different Baxterization ansätze. Based on this observation we find the complete classification of exactly soluble generalized isotropic spin-$\frac{1}{2}$ ladder models.

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1. Introduction

Exactly soluble models provide considerable insight into the complicated behaviour of frustrated quasi-one-dimensional spin systems. Here we investigate the most general spin ladder which includes isotropic Heisenberg leg, rung and diagonal interactions, as well as biquadratic leg–leg, diagonal–diagonal and rung–rung spin exchange terms. The ladder system represents the minimal model admitting both multiple-spin exchange interactions and non-trivial exact solutions.

The spin models on a ladder are usually characterized in terms of their symmetry, i.e., the invariance of the Hamiltonian with respect to transformations in ordinary space and in spin space. In [1] the $su(4)$ and $su(3|1)$-invariant integrable Hamiltonians have been constructed on the basis of the solution of the Yang–Baxter equation (R-matrix), given in terms of the corresponding permutation operator. Multiparametric models related to these symmetries have been considered in [2] and [3] while an extension to n-leg ladder models associated with the $su(2^n)$, $o(2^n)$ and $sp(2^n)$ algebras has been obtained in [4] and [5]. The R-matrix of the generalized ladder model [4, 6] satisfies the Yang–Baxter equation with a multiplicative composition law. Physical properties of some exactly soluble ladder systems, such as magnetization plateaus and thermodynamics, have been presented in [1, 7–10].

From the general point of view, many known examples of solutions of the Yang–Baxter equation are based on algebraic structures, such as braid algebra, Temperley–Lieb and Hecke algebras, Birman–Wenzl–Murakami algebra. Therefore it is natural to search for the algebraic construction suitable for describing $SU(2)$-invariant spin ladders.

We study exactly soluble models within the space of coupling parameters. The general ladder Hamiltonian is based on a plaquette algebra, by which is meant an algebra of operators defined on the four-spin plaquette units of the ladder. This algebra satisfies quasi-local

commutation relations, which means that operators defined on next-to-nearest (and further-) neighbour plaquettes commute. Using the commutation relations of this algebra we find exact expressions for different R-matrices which give solutions to the Yang–Baxter equation. These solutions correspond in turn to soluble (and $SU(2)$-invariant) spin-ladder Hamiltonians, which may be either gapless or gapped.

The plaquette algebra provides a representation of the undeformed limit of a dilute, two-colour braid–monoid algebra. The dilute, two-colour braid–monoid or two-colour Birman–Wenzl–Murakami algebra was introduced and investigated in a series of papers by Grimm and co-workers [11–15]. In particular situations, when the generators of the undeformed dilute algebra are combined into full-braid and monoid generators, the algebra is equivalent to the Brauer [16] or braid–monoid algebra [17], which is the undeformed limit of the Birman–Wenzl–Murakami algebra. By taking all possible different combinations of generators of the algebra, we construct the solutions of the Yang–Baxter equation, and thus the corresponding exactly soluble Hamiltonians. Because the algebra provides a complete basis for $SU(2)$-invariant ladder we believe that the list of isotropic $SU(2)$-invariant integrable ladder models (with short-range interactions) presented here is complete.

In section 2 we introduce the general spin-$\frac{1}{2}$ ladder model and express all isotropic interactions in the plaquette-algebra basis. By seeking commutation relations of this algebra we observe that it provides a representation of the undeformed limit of a dilute two-colour braid–monoid algebra. In section 3 we use the commutation relations of the algebra to establish a number of Baxterization ansätze by considering different subsets of the generators of the algebra. We thus obtain all exactly soluble Hamiltonians in the parameter space of the generalized ladder Hamiltonian. In section 4, we point out the relation between words defined on the plaquette algebra and matrix-product ansatz states. In the concluding section we summarize the results and make contact with known integrable models.

2. The ladder model and its algebraic structure

2.1. Model and operator basis

The Hamiltonian of the two-leg ladder under consideration is

$$H = \sum_i J_L(\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \mathbf{T}_i \cdot \mathbf{T}_{i+1}) + J_R\mathbf{S}_i \cdot \mathbf{T}_i + J_D(\mathbf{S}_i \cdot \mathbf{T}_{i+1} + \mathbf{T}_i \cdot \mathbf{S}_{i+1})$$
$$+ V_{LL}(\mathbf{S}_i \cdot \mathbf{S}_{i+1})(\mathbf{T}_i \cdot \mathbf{T}_{i+1}) + V_{RR}(\mathbf{S}_i \cdot \mathbf{T}_i)(\mathbf{S}_{i+1} \cdot \mathbf{T}_{i+1})$$
$$+ V_{DD}(\mathbf{S}_i \cdot \mathbf{T}_{i+1})(\mathbf{S}_{i+1} \cdot \mathbf{T}_i) \tag{1}$$

where we include leg (J_L), rung (J_R) and diagonal (J_D) Heisenberg interactions, as well as leg–leg (V_{LL}), rung–rung (V_{RR}) and diagonal–diagonal (V_{DD}), four-spin interactions. Spin-$\frac{1}{2}$ operators on rung i are represented, respectively, by $\mathbf{S}_i$ and $\mathbf{T}_i$, $i = 1, \ldots, L$, and periodic boundary conditions are assumed, $\mathbf{S}_{L+1} = \mathbf{S}_1$ and $\mathbf{T}_{L+1} = \mathbf{T}_1$. Many solutions listed in the next section are invariant with respect to the intertwining transformation

$$\mathbf{S}_{2i} \leftrightarrow \mathbf{S}_{2i} \qquad \mathbf{T}_{2i} \leftrightarrow \mathbf{T}_{2i} \qquad \mathbf{T}_{2i+1} \leftrightarrow \mathbf{S}_{2i+1} \qquad J_L \leftrightarrow J_D \qquad V_{LL} \leftrightarrow V_{DD} \tag{2}$$

which interchanges $\mathbf{S}$ and $\mathbf{T}$ on every second rung and therefore we present one representative Hamiltonian for each pair related by this transformation.

We introduce the orthonormal basis

$$|0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \qquad |1\rangle = |\uparrow\uparrow\rangle$$
$$|2\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \qquad |3\rangle = |\downarrow\downarrow\rangle \tag{3}$$

for each rung, and construct a set of corresponding projection operators which generates the $SU(4)$ algebra (hereafter Greek indices denote 0, 1, 2, 3)

$$X_i^{\alpha\beta} = (|\alpha\rangle\langle\beta|)_i \qquad X_i^{\alpha\beta} X_i^{\delta\gamma} = \delta^{\beta\delta} X_i^{\alpha\gamma}$$

$$\sum_\alpha X_i^{\alpha\alpha} = 1 \qquad [X_i^{\alpha\beta}, X_j^{\gamma\delta}] = (\delta^{\beta\gamma} X_i^{\alpha\delta} - \delta^{\alpha\delta} X_j^{\gamma\beta})\delta_{ij}. \tag{4}$$

The spin operators $\mathbf{S}_i$ and $\mathbf{S}_i$ may be expressed in terms of these projection operators. The ladder Hamiltonian (1) is thus equivalent to a four-state chain with only nearest-neighbour interactions.

We introduce the basis of operators

$$b_i = \sum_a X_i^{0a} X_{i+1}^{a0} \qquad b_i^\dagger = \sum_a X_i^{a0} X_{i+1}^{0a} \qquad B_i = \sum_{a,b} X_i^{ab} X_{i+1}^{ba}$$

$$a_i^\dagger = \sum_a (-1)^a X_i^{a0} X_{i+1}^{(4-a)0} \qquad a_i = \sum_b (-1)^b X_i^{0b} X_{i+1}^{0(4-b)}$$

$$e_i = a_i^\dagger a_i = \sum_{a,b} (-1)^{a+b} X_i^{ab} X_{i+1}^{(4-a)(4-b)} \qquad p_i^0 = (1 - X_i^{00})(1 - X_{i+1}^{00}) \tag{5}$$

$$p_i^1 = X_i^{00}(1 - X_{i+1}^{00}) \qquad p_i^2 = (1 - X_i^{00})X_{i+1}^{00} \qquad p_i^3 = X_i^{00} X_{i+1}^{00} \qquad \sum_{\alpha=0}^{3} p_i^\alpha = I_i$$

where Latin indices a, b take the values 1, 2, 3 and we have used the plaquette notation $O_{i,i+1} \equiv O_i$ for all operators.

This basis provides the full set of operators for isotropic interactions, i.e. any $SU(2)$-invariant term may be represented by linear combinations of the above generators. The multiplication table of these operators has the form

	B_i	e_i	b_i	$b_i^\dagger$	a_i	$a_i^\dagger$	p_i^0	p_i^1	p_i^2	p_i^3
B_i	p_i^0	e_i	0	0	0	$a_i^\dagger$	B_i	0	0	0
e_i	e_i	$3e_i$	0	0	0	$3a_i^\dagger$	e_i	0	0	0
b_i	0	0	0	p_i^1	0	0	0	0	b_i	0
$b_i^\dagger$	0	0	p_i^2	0	0	0	0	$b_i^\dagger$	0	0
a_i	a_i	$3a_i$	0	0	0	$3p_i^3$	a_i	0	0	0
$a_i^\dagger$	0	0	0	0	e_i	0	0	0	0	$a_i^\dagger$
p_i^0	B_i	e_i	0	0	0	$a_i^\dagger$	p_i^0	0	0	0
p_i^1	0	0	b_i	0	0	0	0	p_i^1	0	0
p_i^2	0	0	0	$b_i^\dagger$	0	0	0	0	p_i^2	0
p_i^3	0	0	0	0	a_i	0	0	0	0	p_i^3

where in a product of two operators of the form $O_i^{(1)} O_i^{(2)}$, the operator $O_i^{(1)}$ is taken from the column on the left whereas the operator $O_i^{(2)}$ belongs to the top row.

In the operator basis defined by equation (5), the Hamiltonian (1) has the form

$$H = \sum_i g_1 B_i + g_2 e_i + g_3(b_i + b_i^\dagger) + g_4(a_i + a_i^\dagger) + g_5 p_i^0 + g_6 p_i^3 + c \tag{6}$$

where

$$g_1 = \frac{J_L}{2} + \frac{J_D}{2} + \frac{V_{LL}}{8} + \frac{V_{DD}}{8} \qquad g_2 = -\frac{J_L}{2} - \frac{J_D}{2} + \frac{V_{LL}}{8} + \frac{V_{DD}}{8}$$

$$g_3 = \frac{J_L}{2} - \frac{J_D}{2} + \frac{V_{LL}}{8} - \frac{V_{DD}}{8} \qquad g_4 = \frac{J_L}{2} - \frac{J_D}{2} - \frac{V_{LL}}{8} + \frac{V_{DD}}{8}$$

$$g_5 = \frac{V_{RR}}{4} + \frac{J_R}{2} \qquad\qquad\qquad g_6 = -\frac{J_R}{2} + \frac{V_{LL}}{4} + \frac{V_{DD}}{4} + \frac{3V_{RR}}{4}$$

$$c = -\frac{J_R}{4} - \frac{V_{LL}}{16} - \frac{V_{DD}}{16} - \frac{3V_{RR}}{16}.$$

$$(7)$$

2.2. Operator algebra

The operators defined in (5) generate a closed algebra whose commutation relations for operators defined on the same plaquette are given in the multiplication table, while for operators on neighbouring plaquettes one obtains

$$
\begin{aligned}
e_i e_{i+1} e_i &= e_i p_{i+1}^0 & e_{i+1} e_i e_{i+1} &= p_i^0 e_{i+1} \\
B_i B_{i+1} B_i &= B_{i+1} B_i B_{i+1} & e_i B_{i+1} e_i &= e_i p_{i+1}^0 \\
B_i e_{i+1} &= B_{i+1} e_i e_{i+1} & e_i B_{i+1} &= e_i e_{i+1} B_i \\
B_i e_{i+1} B_i &= B_{i+1} e_i B_{i+1}
\end{aligned}
\tag{8}
$$

for operators e_i called monoids and B_i called braid operators, and

$$
\begin{aligned}
b_i^\dagger a_{i+1}^\dagger &= b_{i+1} a_i^\dagger & b_i^\dagger a_{i+1}^\dagger a_i &= b_{i+1} e_i \\
b_i^\dagger b_{i+1}^\dagger &= a_{i+1} a_i^\dagger & b_i^\dagger b_{i+1}^\dagger a_i &= a_{i+1} e_i \\
b_i^\dagger e_{i+1} &= b_{i+1} a_i^\dagger a_{i+1} & b_i^\dagger e_{i+1} b_i &= b_{i+1} e_i b_{i+1}^\dagger \\
b_i^\dagger B_{i+1} b_i &= b_{i+1} B_i b_{i+1}^\dagger
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
b_i b_{i+1} a_i^\dagger &= a_{i+1}^\dagger p_i^3 & b_i a_{i+1} e_i &= b_{i+1}^\dagger a_i \\
b_i a_{i+1} a_i^\dagger &= p_i^1 b_{i+1}^\dagger & b_i a_{i+1} B_i &= b_{i+1}^\dagger a_i B_{i+1} \\
b_i b_{i+1} B_i &= B_{i+1} b_i b_{i+1}
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
a_i^\dagger b_{i+1} b_i &= e_i a_{i+1}^\dagger & a_i^\dagger a_{i+1} b_i &= e_i b_{i+1}^\dagger \\
a_i^\dagger b_{i+1} &= e_i a_{i+1}^\dagger b_i & a_i^\dagger a_{i+1} &= e_i b_{i+1}^\dagger b_i^\dagger
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
a_i a_{i+1}^\dagger &= b_{i+1} b_i & a_i a_{i+1}^\dagger b_i^\dagger &= b_{i+1} p_i^1 \\
a_i b_{i+1}^\dagger &= a_{i+1} b_i & a_i b_{i+1}^\dagger b_i^\dagger &= a_{i+1} p_i^1 \\
a_i e_{i+1} &= a_i B_{i+1} B_i = b_{i+1} b_i a_{i+1} & a_i B_{i+1} e_i &= a_i p_{i+1}^0 \\
a_i e_{i+1} e_i &= a_i B_{i+1} e_i = a_i p_{i+1}^0 & a_i e_{i+1} B_i &= a_i B_{i+1} \\
a_i e_{i+1} a_i^\dagger &= a_i B_{i+1} a_i^\dagger = p_i^3 p_{i+1}^1
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
e_i a_{i+1}^\dagger &= a_i^\dagger b_{i+1} b_i = B_{i+1} B_i a_{i+1}^\dagger & e_i e_{i+1} a_i^\dagger &= a_i^\dagger p_{i+1}^1 \\
e_i a_{i+1}^\dagger b_i^\dagger &= a_i^\dagger b_{i+1} & e_i b_{i+1}^\dagger &= a_i^\dagger a_{i+1} b_i \\
e_i b_{i+1}^\dagger b_i^\dagger &= a_i^\dagger a_{i+1} & e_i B_{i+1} a_i^\dagger &= a_i^\dagger p_{i+1}^1
\end{aligned}
\tag{13}
$$

$$B_i a_{i+1}^\dagger = B_{i+1} e_i a_{i+1}^\dagger \qquad\qquad B_i a_{i+1}^\dagger b_i^\dagger = B_{i+1} a_i^\dagger b_{i+1}$$
$$B_i e_{i+1} a_i^\dagger = B_{i+1} a_i^\dagger \qquad\qquad B_i B_{i+1} a_i^\dagger = e_{i+1} a_i^\dagger \tag{14}$$
$$B_i b_{i+1}^\dagger b_i^\dagger = b_{i+1}^\dagger b_i^\dagger B_{i+1}$$

between operators of different kinds. For $|i - j| > 1$ all operators commute.

This algebra provides a representation of the dilute two-colour braid–monoid algebra introduced and investigated in [11–15], or more explicitly, to its undeformed limit. It emerges in the investigation of dilute A-D-E models and may be considered a special case of the two-colour generalization of the braid–monoid algebra [17], where the representation of one of the colours is trivial. This algebra possesses a graphical interpretation in terms of strands or strings, such that the relations of the algebra are equivalent to the deformation of the diagrams.

The operators e_i satisfy the relations of the Temperley–Lieb algebra [18] (see the first line in equation (8)). The braid generators B_i, together with e_i, satisfy the braid–monoid (or Brauer [16]) algebra (8), a subalgebra of the dilute two-colour braid–monoid algebra. Later we will also introduce full-braid and full monoid generators which satisfy the $O(4)$-related Brauer algebra relations. All of these algebraic relations permit the construction of different Baxterization patterns.

2.3. Baxterization

The Yang–Baxter equation plays a central role in the study of integrable models. By starting from the braid-group representation one may construct its solutions, which depend on the spectral parameter, and demand that in the limit, as the spectral parameter is taken to be zero, the given braid-group representation is recovered. The process of seeking this solution is called Baxterization. It was introduced by Jones in [19], where it was found that the Birman–Wenzl–Murakami algebra provides the solution of Yang–Baxter equation. In the present considerations, algebraic relations (9)–(14) permit the construction of different Baxterization patterns.

The solution of the Yang–Baxter equation

$$\check{R}_i(u)\check{R}_{i+1}(u + v)\check{R}_i(v) = \check{R}_{i+1}(v)\check{R}_i(u + v)\check{R}_{i+1}(u) \tag{15}$$

where u is real, is sought in the form

$$\check{R}_i(u) = a(u)\big(a_i + a_i^\dagger\big) + b(u)\big(b_i + b_i^\dagger\big) + c(u)B_i + d(u)e_i$$
$$+ e(u)p_i^0 + f(u)\big(p_i^1 + p_i^2\big) + g(u)p_i^3. \tag{16}$$

Following the standard analysis we require the initial condition, $\check{R}_i(0) = I_i$, and the unitarity condition, $\check{R}_i(u)\check{R}_i(-u) = k(u)k(-u)I_i$, for some function $k(u)$ which we demand to be rational, trigonometric or hyperbolic (in this work). Following the algebraic Bethe ansatz technique (see e.g. [20] for a review), the corresponding soluble Hamiltonian is constructed as

$$H = \sum_i \frac{\mathrm{d}\check{R}_i(u)}{\mathrm{d}u}\bigg|_{u=0}. \tag{17}$$

This equation amounts to a strong condition on the general Hamiltonian (6) and thus restricts the values of coupling parameters, for which the model is integrable, to a few points and lines in parameter space.

3. Exact solutions

We study different particular cases by taking certain operator combinations. The set of 'basic' generators $\{B_i, e_i, a_i, a_i^\dagger, b_i, b_i^\dagger\}$ forms a basis for all interaction terms in (1) except for rung and rung–rung interactions. The Hermiticity of the Hamiltonian requires that operators $(a_i, a_i^\dagger)$ and $(b_i, b_i^\dagger)$ enter only as the sums $a_i + a_i^\dagger$ and $b_i + b_i^\dagger$. Simple combinatorial arguments give the total possible number of exactly soluble models. Four combinations may be constructed from single generators (i.e., $B_i, e_i, a_i + a_i^\dagger$ and $b_i + b_i^\dagger$), there are six pair combinations, four triple combinations and one combination constructed from all four generators. All of these cases must be considered together with the projectors p_i^α. The combination $a_i + a_i^\dagger$ enters only together with the monoid e_i, as otherwise the unitarity condition for the $\check{R}$ matrix is not satisfied. This excludes one single, two pairs and one triple combination. Thus there are in total 11 possible combinations which may potentially yield exactly soluble models, but we will show that not all of these are relevant to the spin-$\frac{1}{2}$ ladder or possess a Baxterization ansatz. For example, two triple combinations, $\{B_i, b_i + b_i^\dagger, e_i\}$ and $\{B_i, a_i + a_i^\dagger, e_i\}$, appear not to be integrable. In these cases the integrability test [21] for the local Hamiltonian $h_{i,i+1}$,

$$\sum_i [h_{i,i+1} + h_{i+1,i+2}, [h_{i,i+1}, h_{i+1,i+2}]] = 0 \tag{18}$$

is not satisfied, whereas for all the integrable cases investigated in the present work it is satisfied.

For certain cases we will present particular discrete transformations which increase the number of soluble Hamiltonians.

3.1. The combination $\{b_i, b_i^\dagger, p_i^\alpha\}$

This combination gives the solution to the Yang–Baxter equation for which the corresponding R-matrix has the form

$$\check{R}_i(u) = I_i + u\big(b_i + b_i^\dagger + p_i^0 + p_i^3\big). \tag{19}$$

This solution has exactly the form of the spin-$\frac{1}{2}$ Heisenberg model. Indeed, the composite operator $\Pi_i = b_i + b_i^\dagger + p_i^0 + p_i^3$ satisfies the braid algebra relation $\Pi_i \Pi_{i+1} \Pi_i = \Pi_{i+1} \Pi_i \Pi_{i+1}$ and its square gives the identity operator. The Hamiltonian which follows from this R-matrix has exactly the same eigenspectrum as the spin-$\frac{1}{2}$ Heisenberg model, but the degeneracy of this spectrum is different [22]. The corresponding Hamiltonian $H_{1/2} = \sum_i \Pi_i$ commutes with the generators $\sum_i X_i^{ab}$ for any $a, b = 1, 2, 3$. These generators form an $su(3)$ subalgebra within the $su(4)$ algebra generated by the X-operators. $H_{1/2}$ commutes with the operator $\sum_i X_i^{00}$, which represents the total number of rung singlets, and any multiple of this term may be added to the Hamiltonian without affecting the integrability. It corresponds to a coupling of the effective spin-$\frac{1}{2}$ Heisenberg model to an effective 'magnetic field' h. It is well known that the spin-$\frac{1}{2}$ chain in a magnetic field develops an incommensurate critical phase for $|h| \leqslant 2$ and has a massive phase for $|h| > 2$. Returning to the original spin variables, we have

$$H_{1/2} = \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \mathbf{T}_i \cdot \mathbf{T}_{i+1} - \mathbf{S}_i \cdot \mathbf{T}_{i+1} - \mathbf{T}_i \cdot \mathbf{S}_{i+1} + 4[(\mathbf{S}_i \cdot \mathbf{S}_{i+1})(\mathbf{T}_i \cdot \mathbf{T}_{i+1})$$
$$+ (\mathbf{S}_i \cdot \mathbf{T}_i)(\mathbf{S}_{i+1} \cdot \mathbf{T}_{i+1}) - (\mathbf{S}_i \cdot \mathbf{T}_{i+1})(\mathbf{S}_{i+1} \cdot \mathbf{T}_i)] + (J_R + 2)(\mathbf{S}_i \cdot \mathbf{T}_i), \tag{20}$$

where $J_R = 0$ corresponds to the Hamiltonian derived from the $\check{R}$-matrix (19). The Hamiltonian commutes with $\sum_i X_i^{00}$ and remains therefore integrable for arbitrary J_R. For $-4 \leqslant J_R \leqslant 4$ the model is in the critical, incommensurate phase, while for $J_R > 4$ it has a

gapped rung-singlet phase, and in the region $J_R < -4$ it has a rung-triplet ground state. It coincides with the model studied in [22] and more recently in [6]. Although the $\check{R}$-matrix of [6] is not the same as that in (19), we believe that they are related by an appropriate transformation.

In the critical region the model is described by a conformal field theory with central charge $c = 1$. However, because of the degeneracy this conformal field theory possesses additional zero modes, and therefore the conformal dimensions appearing in this model have the Coulomb-gas dimensions of the $c = 1$ theory, but with different degeneracies.

3.2. The combination $\left\{b_i, b_i^\dagger, B_i, p_i^\alpha\right\}$

This combination leads to the $\check{R}$-matrix

$$\check{R}_i(u) = I_i + u\left(\pm\left(b_i + b_i^\dagger\right) + B_i + p_i^3\right) \tag{21}$$

a solution which corresponds simply to local Hamiltonian operators proportional to the permutation operators (see section 3.5 below) $P_i^{\parallel}$ (for positive sign) and $P_i^{\times}$ (for negative sign). These Hamiltonians are invariant with respect to the full $SU(4)$ group, and describe the soluble 'spin-orbital' model [23]

$$H_{P\parallel} = \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \mathbf{T}_i \cdot \mathbf{T}_{i+1} + 4(\mathbf{S}_i \cdot \mathbf{S}_{i+1})(\mathbf{T}_i \cdot \mathbf{T}_{i+1}). \tag{22}$$

The model corresponding to the $P_i^{\times}$ can be obtained from (22) by applying the transformation (2). Both Hamiltonians commute with $\sum_i X_i^{00}$, and therefore the term $J_R \sum_i \mathbf{S}_i \cdot \mathbf{T}_i$ may be added to these Hamiltonians without affecting the property of integrability. The model (22) was studied in [1] for certain values of the rung–rung interaction parameter V_{RR} added to the Hamiltonian. For some regions in the parameter space of J_R and V_{RR}, these models demonstrate a variety of gapless behaviour. In particular, the model (22) in the continuum limit corresponds to the $SU(4)$ Wess–Zumino–Witten (WZW) model at level 1 of the Kac–Moody algebra [24, 25]. The central charge is equal to 3.

3.3. The combination $\left\{b_i, b_i^\dagger, e_i, p_i^\alpha\right\}$

This yields the hyperbolic solution

$$\check{R}_i(u) = \frac{\sinh(\lambda - u)}{\sinh(\lambda)}\left(p_i^0 + p_i^3\right) + \left(p_i^1 + p_i^2\right) + \frac{\sinh(u)}{\sinh(\lambda)}\left(e_i + \sigma\left(b_i + b_i^\dagger\right)\right) \tag{23}$$

with $\cosh(\lambda) = 3/2$ and $\sigma = \pm 1$.

The corresponding Hamiltonian is

$$H = \sum_i -\mathbf{S}_i \cdot \mathbf{T}_{i+1} - \mathbf{T}_i \cdot \mathbf{S}_{i+1} + 4[(\mathbf{S}_i \cdot \mathbf{S}_{i+1})(\mathbf{T}_i \cdot \mathbf{T}_{i+1})$$
$$- (\mathbf{S}_i \cdot \mathbf{T}_i)(\mathbf{S}_{i+1} \cdot \mathbf{T}_{i+1})] + J_R(\mathbf{S}_i \cdot \mathbf{T}_i) \tag{24}$$

for $\sigma = +1$ and $J_R = -1$, and the case of $\sigma = -1$ is obtained by the transformation (2). We note that these Hamiltonians commute with the rung singlet number operator $\sum_i X_i^{00}$, and that the model is therefore integrable for any J_R.

3.4. The combination $\{b_i, b_i^\dagger, e_i, a_i, a_i^\dagger, p_i^\alpha\}$

This combination also permits Baxterization. The solution is given by the Izergin–Korepin model, which is related to the dilute $A_2^{(2)}$ model [26, 27],

$$
\check{R}_i(u) = \frac{\sin(2\lambda - u)\sin(3\lambda - u)}{\sin(2\lambda)\sin(3\lambda)} p_i^0 + \frac{\sin(3\lambda - u)}{\sin(3\lambda)}(p_i^1 + p_i^2) + \left(1 + \frac{\sin(u)\sin(3\lambda - u)}{\sin(2\lambda)\sin(3\lambda)}\right) p_i^3
$$
$$
+ \frac{\sin(u)\sin(3\lambda - u)}{\sin(2\lambda)\sin(3\lambda)}(b_i^\dagger + b_i) - \frac{\sin(u)\sin(\lambda - u)}{\sin(2\lambda)\sin(3\lambda)} e_i + \frac{\sin(u)}{\sin(3\lambda)}(a_i^\dagger + a_i).
$$

$$(25)$$

However, the value of the Temperley–Lieb factor (or fugacity in corresponding statistical mechanics models) n in the relation $e_i^2 = ne_i$ is restricted here by the condition $n = -2\cos(4\lambda)$. In our case $n = 3$ (see multiplication table) and therefore λ is necessarily complex. The analytic continuation of the weights into the domain $|n| > 2$ by the change $\lambda \to \pi/4 + i\tilde{\lambda}$ and $u \to i\tilde{u}$ makes these complex, and the corresponding Hamiltonian then has complex coefficients which make this operator combination not relevant for the spin-$\frac{1}{2}$ ladder model.

3.5. The full set of operators

The full set of operators provides solutions related to the Brauer algebra [16], or undeformed braid–monoid algebra [17]. In order to present this solution we introduce full braid and monoid operators, whose relations with the generators (5) of the dilute algebra are given by

$$
\begin{array}{ll}
E_i^{(-)} = e_i + a_i + a_i^\dagger + p_i^3 & E_i^{(+)} = e_i - a_i - a_i^\dagger + p_i^3 \\
P_i^\| = B_i + b_i + b_i^\dagger + p_i^3 & P_i^\times = B_i - b_i - b_i^\dagger + p_i^3
\end{array}
$$

$$(26)$$

where $P_{i,i+1}^\|$ and $P_{i,i+1}^\times$ are, respectively, the permutation operators corresponding to leg–leg and diagonal–diagonal bonds.

The operators $E_{i,i+1}^{(-)}$, $E_{i,i+1}^{(+)}$, $P_{i,i+1}^\|$ and $P_{i,i+1}^\times$ are generators of the algebra

$$
\begin{array}{lll}
E_i^2 = 4E_i & E_i E_{i+1} E_i = E_i & E_{i+1} E_i E_{i+1} = E_{i+1} \\
P_i^2 = 1 & P_i P_{i+1} P_i = P_{i+1} P_i P_{i+1} & P_i E_i = E_i P_i = E_i \\
P_i E_{i+1} E_i = P_{i+1} E_i & E_{i+1} E_i P_{i+1} = E_{i+1} P_i & P_i E_{i+1} P_i = P_{i+1} E_i P_{i+1}
\end{array}
$$

and

$$
\begin{array}{llll}
P_i P_j = P_j P_i & E_i E_j = E_j E_i & P_i E_j = E_j P_i & \text{for} \quad |i - j| > 1.
\end{array}
$$

$$(27)$$

We have again used the plaquette notation $P_{i,i+1} \equiv P_i$ and $E_{i,i+1} \equiv E_i$, where P_i denotes either $P_i^\|$ or $P_i^\times$ and E_i denotes either $E_i^{(-)}$ or $E_i^{(+)}$. Note that there exist a number of relations between the operators, which follow from the above. The operators P_i satisfy the braid algebra relations, while the operators E_i form a Temperley–Lieb algebra [18]. Together these operators form a Brauer [16] (or braid–monoid [17]) algebra. The operators $E_i^{(\pm)}$ are projectors on the plaquette singlet states (below).

This algebra allows Baxterization [17, 28], the solution for the case under consideration being given by the rational $\check{R}$ matrix

$$
\check{R}_i(u) = I_i + uP_i - \frac{u}{u + 1} E_i
$$

$$(28)$$

which satisfies the initial and unitarity conditions with $k(u) = 1 + u$. The spin Hamiltonian which corresponds to this R matrix is

$$H = \sum_i (P_i - E_i). \tag{29}$$

We thus obtain four different soluble Hamiltonians which correspond to the four combinations $\left\{ P_i^{\parallel}, E_i^{(-)} \right\}$, $\left\{ P_i^{\parallel}, E_i^{(+)} \right\}$, $\left\{ P_i^{\times}, E_i^{(-)} \right\}$ and $\left\{ P_i^{\times}, E_i^{(+)} \right\}$. Two of these are trivially soluble, because $\sum_i \left(P_i^{\parallel} - E_i^{(+)} \right)$ is the Hamiltonian of two decoupled chains and $\sum_i \left(P_i^{\times} - E_i^{(-)} \right)$ is the same pair of chains intertwined by the transformation (2). Of the remaining two, one is given by the combination $\left\{ P_i^{\parallel}, E_i^{(-)} \right\}$, which in terms of spin operators is

$$H_B = \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} + \mathbf{T}_i \cdot \mathbf{T}_{i+1} + \mathbf{S}_i \cdot \mathbf{T}_{i+1} + \mathbf{T}_i \cdot \mathbf{S}_{i+1}$$

$$+ 4[(\mathbf{S}_i \cdot \mathbf{S}_{i+1})(\mathbf{T}_i \cdot \mathbf{T}_{i+1}) - (\mathbf{S}_i \cdot \mathbf{T}_{i+1})(\mathbf{S}_{i+1} \cdot \mathbf{T}_i)]. \tag{30}$$

The last, $\{ P_i^{\times}, E_i^{(+)} \}$, is obtained from the previous Hamiltonian by the same intertwining transformation (2).

The Brauer algebra (27) is related to the representations of the group $D_2 = O(4)$, and therefore the corresponding Bethe ansatz is also related to this algebra. In fact the corresponding operator $\mathcal{L}$ is the product of two $\mathcal{L}$-operators for six-vertex models. The eigenvalues of the algebraic Bethe ansatz are given by the product of two sets of eigenvalues corresponding to six-vertex models [29], and finally one obtains two decoupled Bethe ansatz equations which correspond to the $D_2 = SU(2) \times SU(2)$ algebra. The corresponding energy spectrum is the sum of the eigenvalues for each $SU(2)$ component. Thus the Hamiltonian (30) is equivalent to the Hamiltonian of two decoupled chains. This model is critical (no spin gap) with a central charge $c = 2$.

The two projectors $E_i^{(-)}$ and $E_i^{(+)}$ are related by the X_i-operator transformation

$$X_i^{0a} \to -\mathrm{i} X_i^{0a} \qquad X_i^{a0} \to \mathrm{i} X_i^{a0} \tag{31}$$

for all sites $i = 1, \ldots, N$. This is a unitary transformation generated by the operator

$$U(\pi/2) = \exp\left[-\mathrm{i}\frac{\pi}{2} \sum_{i=1}^N \left(X_i^{00} \right) \right]. \tag{32}$$

It leaves P_i invariant and transforms the Hamiltonian of two decoupled chains into H_B. Moreover, because of the relation $[U(\pi/2)]^4 = 1$, this transformation is one of the generators of the Z_4 group.

The projectors may be decomposed into the product of two operators, namely

$$E_i^{(-)} = A_i^{\dagger} A_i \qquad E_i^{(+)} = B_i^{\dagger} B_i \tag{33}$$

where

$$A_i^{\dagger} = X_i^{00} X_{i+1}^{00} + X_i^{20} X_{i+1}^{20} - X_i^{10} X_{i+1}^{30} - X_i^{30} X_{i+1}^{10}$$

$$B_i^{\dagger} = X_i^{00} X_{i+1}^{00} - X_i^{20} X_{i+1}^{20} + X_i^{10} X_{i+1}^{30} + X_i^{30} X_{i+1}^{10}. \tag{34}$$

These operators obey the relations

$$A_i^2 = A_i \qquad\qquad B_i^2 = B_i$$

$$A_i A_i^{\dagger} = 4 X_i^{00} X_{i+1}^{00} \qquad B_i B_i^{\dagger} = 4 X_i^{00} X_{i+1}^{00}. \tag{35}$$

The transformation (32) yields $A_i \leftrightarrow B_i$ and $E_i^{(-)} \leftrightarrow E_i^{(+)}$. The unnormalized projectors may be expressed as

$$E_i^{(-)} = \left| \psi_i^{(-)} \right\rangle\!\left\langle \psi_i^{(-)} \right| \qquad E_i^{(+)} = \left| \psi_i^{(+)} \right\rangle\!\left\langle \psi_i^{(+)} \right| \tag{36}$$

where

$$\left|\psi_i^{(-)}\right\rangle = A_i^\dagger|0\rangle \qquad \left|\psi_i^{(+)}\right\rangle = B_i^\dagger|0\rangle \tag{37}$$

and we have defined the vacuum state as a direct product of singlet states for all rungs. The states $\left|\psi_i^{(-)}\right\rangle$ and $\left|\psi_i^{(+)}\right\rangle$ may be expressed in terms of local singlet states $|s_{ij}\rangle$ ($i, j = 1, 2, 3, 4$) on the bonds of each plaquette. Using the X-operator notation and denoting the plaquette vertices in clockwise order, one may write

$$\left|\psi_i^{(-)}\right\rangle = -2(|s_{12}\rangle|s_{34}\rangle - |s_{23}\rangle|s_{41}\rangle) \qquad \left|\psi_i^{(+)}\right\rangle = -2|s_{23}\rangle|s_{41}\rangle. \tag{38}$$

These functions yield a plaquette-singlet state, i.e. the state with the total spin per plaquette $(\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4)^2$ equal to zero. There are exactly two possible plaquette-singlet states which correspond, respectively, to $E_i^{(-)}$ and $E_i^{(+)}$. The Z_4 transformation (32) maps one state onto the other.

In terms of the original spin variables this transformation is given by [30]

$$\tilde{\mathbf{S}}_i = \tfrac{1}{2}(\mathbf{S}_i + \mathbf{T}_i) - \mathbf{S}_i \times \mathbf{T}_i \qquad \tilde{\mathbf{T}}_i = \tfrac{1}{2}(\mathbf{S}_i + \mathbf{T}_i) + \mathbf{S}_i \times \mathbf{T}_i. \tag{39}$$

The transformation is canonical for the spin-$\tfrac{1}{2}$ ⊗ spin-$\tfrac{1}{2}$ representation of $SU(2) \times SU(2)$ group and the values of the Casimir operators are the same, $\tilde{\mathbf{S}}_i^2 = \tilde{\mathbf{T}}_i^2 = 3/4$. In terms of the variables $\tilde{\mathbf{S}}_i$, $\tilde{\mathbf{T}}_i$ the Hamiltonian H_B (30) indeed has the form of two decoupled spin chains.

3.6. The full monoid generators $E_i^{(-)}$ and $E_i^{(+)}$

These combinations allow Baxterization according to the ansatz for the Temperley–Lieb algebra,

$$\check{R}_i(u) = \frac{\sinh(\lambda - u)}{\sinh\lambda} I_i + \frac{\sinh(u)}{\sinh\lambda} E_i \tag{40}$$

where $\cosh(\lambda) = 2$. Here E_i denotes either $E_i^{(-)}$ or $E_i^{(+)}$ defined in equation (26). The corresponding Hamiltonians are

$$H = -\sum_i \{\mathbf{S}_i \cdot \mathbf{T}_{i+1} + \mathbf{T}_i \cdot \mathbf{S}_{i+1} - 4(\mathbf{S}_i \cdot \mathbf{T}_{i+1})(\mathbf{S}_{i+1} \cdot \mathbf{T}_i)\} \tag{41}$$

for the full Temperley–Lieb generator $E_i^{(-)}$, and

$$H = -\sum_i \{\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \mathbf{T}_i \cdot \mathbf{T}_{i+1} - 4(\mathbf{S}_i \cdot \mathbf{S}_{i+1})(\mathbf{T}_i \cdot \mathbf{T}_{i+1})\} \tag{42}$$

for generator $E_i^{(+)}$. We note that these Hamiltonian operators are related by a Z_4 transformation. In this particular case two Hamiltonians (41) and (42) may also be related by the transformation (2). The spectrum of both Temperley–Lieb soluble models is gapped and their ground-state energy and gap may in principle be calculated by a mapping onto the Potts model [31].

3.7. Models related to the bilinear–biquadratic spin-1 chain

In this subsection we project the space of states onto a three-state subspace by applying the projection operator p_i^0 on every site. This gives three soluble models which are related to the generalized spin-1 chain. The role of unit operator I_i is played by the operator p_i^0. The realization of these models in the spin-$\tfrac{1}{2}$ ladder is discussed below.

3.7.1. The combination $\{B_i, e_i, p_i^0\}$. This combination of the braids and monoids of the dilute algebra (5) form the Brauer algebra (27) in a restricted three-state subspace with Temperley–Lieb (fugacity) factor $n = 3$ (in the relation $e_i^2 = ne_i$), and leads to the solution

$$\check{R}_i(u) = I_i^{(3)} + uB_i - \frac{u}{u + 1/2}e_i. \tag{43}$$

This $\check{R}$-matrix solution corresponds to the Takhtajan–Babudjian spin-1 integrable model [32]. The corresponding Hamiltonian is

$$H_{TB} = \sum_i \frac{3}{2}(\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \mathbf{T}_i \cdot \mathbf{T}_{i+1} + \mathbf{S}_i \cdot \mathbf{T}_{i+1} + \mathbf{T}_i \cdot \mathbf{S}_{i+1})$$
$$- 2[(\mathbf{S}_i \cdot \mathbf{S}_{i+1})(\mathbf{T}_i \cdot \mathbf{T}_{i+1}) + (\mathbf{S}_i \cdot \mathbf{T}_{i+1})(\mathbf{S}_{i+1} \cdot \mathbf{T}_i)]. \tag{44}$$

This model is described by a $c = 3/2$, $SU(2)_{k=2}$ WZW model in the continuum limit.

3.7.2. The dilute braid generator B_i. This generator is a permutation operator in the three-state subspace and leads to the $SU(3)$-invariant Uimin–Lai–Sutherland [33] model,

$$\check{R}_i(u) = I_i^{(3)} + uB_i. \tag{45}$$

The spin Hamiltonian is

$$H_{ULS} = \sum_i \frac{1}{2}[\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \mathbf{T}_i \cdot \mathbf{T}_{i+1} + \mathbf{S}_i \cdot \mathbf{T}_{i+1} + \mathbf{T}_i \cdot \mathbf{S}_{i+1}]$$
$$+ 2[(\mathbf{S}_i \cdot \mathbf{S}_{i+1})(\mathbf{T}_i \cdot \mathbf{T}_{i+1}) + (\mathbf{S}_i \cdot \mathbf{T}_{i+1})(\mathbf{S}_{i+1} \cdot \mathbf{T}_i)]. \tag{46}$$

In the continuum limit it is described by the $SU(3)_{k=1}$ WZW model [34].

3.7.3. The dilute monoid operator e_i. This operator generates the Temperley–Lieb algebra in the three-state restricted subspace (see relations (5) and (13)), and leads to the solution given by the usual Baxterization ansatz for the Temperley–Lieb algebra,

$$\check{R}_i(u) = \frac{\sinh(\lambda - u)}{\sinh \lambda} I_i^{(3)} + \frac{\sinh(u)}{\sinh \lambda} e_i \tag{47}$$

where $\cosh(\lambda) = 3/2$. The corresponding Hamiltonian

$$H = \frac{1}{2} \sum_i -\mathbf{S}_i \cdot \mathbf{S}_{i+1} - \mathbf{T}_i \cdot \mathbf{T}_{i+1} - \mathbf{S}_i \cdot \mathbf{T}_{i+1} - \mathbf{T}_i \cdot \mathbf{S}_{i+1}$$
$$+ 4[(\mathbf{S}_i \cdot \mathbf{S}_{i+1})(\mathbf{T}_i \cdot \mathbf{T}_{i+1}) + (\mathbf{S}_i \cdot \mathbf{T}_{i+1})(\mathbf{S}_{i+1} \cdot \mathbf{T}_i)] \tag{48}$$

is related to the spin-1 chain model studied in [31].

3.7.4. Generalization. The above three solutions correspond to soluble, spin-1 and bilinear–biquadratic chains. Their relation to the ladder (with four states per rung) is the following: on rung i the effective spin-1 variable is formed by the generators X_i^{a0} ($a = 1, 2, 3$). If there is a finite region on the ladder formed by rung triplets bounded by rung singlets, $X_{i-1}^{00} X_i^{a0} X_{i+1}^{b0} \cdots X_{k+i}^{c0} X_{k+i+1}^{00}$, one may consider this region as an effective spin-1 chain with open boundary conditions. The three ladder Hamiltonians of this subsection thus yield exactly soluble models for such configuration.

An explicit relationship between the original spin-$\frac{1}{2}$ variables and effective spin-1 operators can also be given by the composite-spin representation [35]. This approach was used in [36].

Considering the ladder with alternating domains of rung triplets and rung singlets, one notes that the couplings V_{RR} and J_R are arbitrary, because the Hamiltonian commutes not only with $\sum_i X_i^{00}$ but also with X_i^{00} for any i, and therefore the eigenstates of the corresponding ladder Hamiltonian in 4^N-dimensional state space are characterized by the value of the total spin at each rung of the ladder. One may study the dependence of the ground-state energy as a function of J_R, V_{RR} and the total number of rung-singlet bonds. As shown in [36] depending on these parameters there exist several gapped phases in such a ladder.

4. Ground state and relation with matrix-product ansatz

We mention further possible implementations of the dilute algebra used in this analysis. The matrix-product ansatz was proved in [37] to be a valuable technique for the description of the gapped states of spin systems. It is based on a decomposition of the spin Hamiltonian into the sum of projection operators on plaquette states with fixed angular momentum. There are two plaquette states with total momentum equal to 0 ($j = 0$), three triplet states ($j = 1$) and one quintuplet ($j = 2$). For particular combinations of coupling constants, it is possible to find exact ground states. In the current formalism, states with $j = 0$ are created by linear combinations of the generators p_i^3 and $a_i^\dagger$ acting on the product of rung-singlet states, states with $j = 1$ are created by the operators $b_i \pm b_i^\dagger$, and by the combinations $B_i - p_i^0$ and $p^1 \pm p^2$, and the state with $j = 2$ is given by $6(B_i + p_i^0 - 4e_i)$. These states have the form of the matrix-product ansatz [37].

A natural interpretation also exists for these states as 'words', the letters of which are elements of the dilute algebra defined by the operators in equation (5). The relation between exact ground states and words of a (Temperley–Lieb) algebra was noted first in [38]. It provides a direct algorithm to search for possible exact ground states: one first classifies all possible simple words of the type $K_{i-1}L_i M_{i+1}$, where in this analysis the operators K, L, M may be any of $a_i, a_i^\dagger, b_i, b_i^\dagger, B_i$ and p_i^α, and then constructs a product of these elementary words. If the Hamiltonian leaves this product invariant, we have obtained an exact eigenstate. In general this procedure can be used for constructing variational wavefunctions. Examples of such states are given by the following words:

$$|\psi_{RS}\rangle \longrightarrow p_1^3 p_2^3 \cdots p_{N-1}^3 p_N^3 \qquad \text{and} \qquad |\psi_{FM}\rangle \longrightarrow p_1^0 p_2^0 \cdots p_{N-1}^0 p_N^0 \tag{49}$$

which correspond to the rung-singlet and ferromagnetic ground states respectively, and

$$|\psi_{j=0}\rangle = \prod_{i=1}^{N} |\psi_{s,j=0}^-\rangle_i \tag{50}$$

where the local state

$$|\psi_{s,j=0}^-\rangle_i = \frac{1}{(1+3x^2)^{1/2}}(p_i^3 - xa_i^\dagger)|0\rangle \tag{51}$$

produces the variational wavefunction which interpolates between the rung-singlet state (for $x = 0$) and one of the plaquette-singlet states (for $x = 1$).

5. Conclusion

We have shown that the undeformed dilute two-colour braid–monoid algebra provides a natural tool for the description of integrable spin-$\frac{1}{2}$ isotropic ladder models with isotropic nearest and next-nearest-neighbour interactions. The Baxterization of different subsets of generators of the algebra yields corresponding spin Hamiltonians that are non-trivial. Their general feature

is the presence of multiple-spin exchange, which may in some cases lead to critical behaviour. The 'words' constructed as a sequence of the elements of the dilute algebra yield variational wavefunctions of the ground state.

At this point it is worthwhile to compare the present work to related studies of integrable ladder models. In fact, some of the models obtained above are not new. For example the spin-orbital type model in section 3.2 is known from the work [1] on soluble $SU(4)$-invariant spin ladders. The class of models related to the integrable bilinear–biquadratic spin-1 chain (section 3.7) has also been constructed using the composite-spin representation [35, 36]. A spin Hamiltonian, similar to equation (20) has already appeared in [6], but the corresponding R-matrix is different. We believe that the connection between these two different R-matrices can be understood on the basis of Hecke-soluble models of [22]. Additional support of this argument is given by [4].

In contrast to the models in sections 3.1, 3.2 and 3.7, which have appeared before in the literature, those in sections 3.3, 3.5 and 3.6 appear to be new. Recently, Batchelor and co-workers [5] have used operator algebras for constructing soluble n-leg spin ladders in terms of $o(2^n)$ and $sp(2^n)$-invariant R-matrices based on Temperley–Lieb and Birman–Wenzl–Murakami algebras. But their construction of algebra operators is different from ours. For the two-leg ladder their Temperley–Lieb generator Q has a fugacity factor -4 ($Q^2 = -4Q$), while our generator E has a fugacity factor equal to $+4$ (see sections 3.5 and 3.6). Correspondingly, the resulting spin Hamiltonians are also different. We also note that although the Hamiltonian (30) may be obtained by a Z_4-transformation from the Hamiltonian of two decoupled spin-$\frac{1}{2}$ chains, the algebraic structure underlying its integrability was not known before.

Applications of these exactly soluble models to real spin ladders may be made by the use of field-theoretical methods. The exact solutions which describe gapless points in parameter space correspond to certain conformal field theories, WZW models, in the continuum limit. The perturbation of these theories by relevant and marginal operators which correspond to different terms in a real spin Hamiltonian produce in general a renormalization-group flow away from the gapless behaviour, thus allowing the determination of the properties of the relevant phases in the vicinity of the points of second-order phase transitions. This programme is realized in the context of real ladder systems with cyclic four spin-exchange interactions in [39].

References

[1] Wang Y 1999 *Phys. Rev.* B **60** 9236–9
[2] Foerster A, Links J and Tonel A P 2000 *Preprint* cond-mat/0005453
[3] Tonel A P, Foerster A, Hibberd K and Links J 2001 *Preprint* cond-mat/0105302
[4] Batchelor M T and Maslen M 1999 *J. Phys. A: Math. Gen.* **32** L377
[5] Batchelor M T, de Gier J, Links J and Maslen M 2000 *J. Phys. A: Math. Gen.* **33** L97
[6] Albeverio S, Fei S M and Wang Y P 1999 *Europhys. Lett.* **47** 364–70
[7] de Gier J and Batchelor M T 2000 *Phys. Rev.* B **62** R3584
[8] Tonel A P, Dahmen S R, Foerster A and Malvezzi A L 2003 *Preprint* cond-mat/0302336

 [9] Batchelor M T, Guan X-W, Foerster A, Tonel A P and Zhou H-Q 2003 *Preprint* cond-mat/0305196
[10] Maslen M, Batchelor M T and de Gier J 2003 *Preprint* cond-mat/0302135
[11] Grimm U and Pearce P A 1993 *J. Phys. A: Math. Gen.* **26** 7435–59
[12] Grimm U 1994 *J. Phys. A: Math. Gen.* **27** 5897–906
[13] Grimm U 1994 *Lett. Math. Phys.* **32** 183–8
[14] Grimm U and Warnaar S Ole 1995 *Nucl. Phys.* B **435** 482–504
[15] Grimm U and Warnaar S Ole 1995 *J. Phys. A: Math. Gen.* **28** 7197–208
[16] Brauer R 1937 *Ann. Math.* **38** 857–72
[17] Wadati M, Deguchi T and Akutsu Y 1989 *Phys. Rep.* **180** 247–332
[18] Temperley H N V and Lieb E H 1971 *Proc. R. Soc.* A **322** 251–80
[19] Jones V 1989 *Commun. Math. Phys.* **125** 459–67
[20] Faddeev L D 1995 *Int. J. Mod. Phys.* A **10** 1845–78
[21] Grabowski M P and Mathieu P 1995 *J. Phys. A: Math. Gen.* **28** 4777–98
[22] Alcaraz F C, Arnaudon D, Rittenberg V and Scheunert M 1994 *Int. J. Mod. Phys.* A **9** 3473
[23] Kugel I and Khomskii D I 1973 *Sov. Phys.—JETP Lett.* **37** 725
[24] Azaria P, Gogolin A O, Lecheminant P and Nersesyan A A 1999 *Phys. Rev. Lett.* **83** 624
[25] Itoi C, Qin S and Affleck I 2000 *Phys. Rev.* **61** 6747
[26] Izergin A G and Korepin V E 1981 *Commun. Math. Phys.* **79** 303
[27] Warnaar S O, Nienhuis B and Seaton K A 1992 *Phys. Rev. Lett.* **69** 710–2
[28] Cheng Y, Ge M L and Xue K 1991 *Commun. Math. Phys.* **136** 195–208
[29] Martins M J and Ramos P B 1997 *Nucl. Phys.* B **500** 579–620
[30] Hikihara T, Momoi T and Hu X 2003 *Phys. Rev. Lett.* **90** 087204
[31] Barber M N and Batchelor M T 1989 *Phys. Rev.* B **40** 4621–6
      Batchelor M T and Barber M N 1990 *J. Phys. A: Math. Gen.* **23** L15
[32] Takhtajan L A 1982 *Phys. Lett.* A **87** 479–82
      Babujian H M 1982 *Phys. Lett.* A **90** 479–82
[33] Uimin G V 1970 *JETP Lett.* **12** 225
      Lai C K 1974 *J. Math. Phys.* **15** 1675
      Sutherland B 1975 *Phys. Rev.* B **12** 3795–805
[34] Affleck I 1986 *Nucl. Phys.* B **265** 409–47
[35] Legeza Ö, Fáth G and Sólyom J 1997 *Phys. Rev.* B **55** 291–8
[36] Albertini G 2001 *Phys. Rev.* B **64** 094416
      Wang Y 1999 *Int. J. Mod. Phys.* B **13** 3323–9
[37] Kolezhuk A K and Mikeska H-J 1998 *Int. J. Mod. Phys.* B **12** 2325–48
[38] Arndt P F, Heinzl T and Yung C M 1995 *J. Phys. A: Math. Gen.* **28** L323–L330
[39] Gritsev V, Normand B and Baeriswyl D unpublished